ION DISTRIBUTION FUNCTION AT THE BOUNDARY WITH AN ELECTRODE

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The ion distribution function is found in the case in which the Langmuir layer freely passes the ions incident from the plasma while the reverse ion flux is zero. These conditions are realized near the cathode in an arc discharge and at the surface of a probe operating on the ion branch of the characteristic. The electric field outside the Langmuir layer is assumed small. We obtain the connection between ion current and plasma density at the boundary with the electrode, the expressions for the ion mean kinetic energy and for the mean energy removed from the plasma by the ion, which differ markedly from the corresponding expressions in the Maxwellian distribution case.

As is known, at the boundary of the plasma with the electrode there arises the Langmuir layer, in which the electric field intensity can be very large. In many important cases the potential drop $e\Delta\varphi$ in the Langmuir layer is considerably larger than the electron T_e or ion T temperatures and is directed so that the electrons flowing from the plasma to the electrode are retarded by the field. This occurs, as an example, near an arc cathode or near a probe when the latter operates on the ion branch of the characteristic. The ions leaving the plasma pass freely through the Langmuir layer and recombine on the electrode surface.

The reverse ion flux from the electrode into the plasma may be assumed zero in the cases considered. In fact, first, the surface ionization which gives rise to this flux may not exist and, secondly, the ions which occur on the wall as a result of surface ionization are retarded in the Langmuir layer and in the case of a large potential drop ($e\Delta\varphi \gg T$) the overwhelming majority of them cannot escape into the plasma. Moreover, in the arc regime, volume ionization is so intensive that surface ionization can usually be neglected in comparison.

In the majority of the important cases we can assume that the thickness of the Langmuir layer is negligibly small (in comparison with the mean free path) and we can consider its effect only in the boundary conditions. The electric field in the Langmuir layer leads to reflection of the electrons incident from the plasma, and therefore their distribution function practically coincides with the Maxwellian function, differing from it only in the high-energy region($\sim e\Delta\varphi \gg T_e$), where the reflection condition does not hold. On the other hand, the ion distribution function may differ significantly from the Maxwellian function, since there is no reverse ion flux from the electrode.

In the following we find the ion distribution at the boundary with the electrode, which is the primary objective of the present study.

1. As a rule, the degree of ionization near electrodes is small. Therefore we must first of all consider scattering of the ions by the neutral atoms. Thus, for a low-voltage arc in cesium at p = 2 torr the ion mean free path in the atoms is about 10μ, while the Coulomb mean free path for the plasma density near the electrode of $1\text{-}2 \cdot 10^{13}$ cm^{-3} and temperature 0.2 eV amounts to $50\text{-}100\,\mu$.

In view of the low degree of ionization the influence of the ions on the atom distribution function can be neglected. On the same basis we can assume that the atom flux incident on the wall is practically equal to the reflected flux. Therefore the atom distribution function must coincide with the Maxwellian distribution function with temperature equal to the wall temperature, while over the distances in question the atom

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temperature remains practically constant. Thus, the atoms play the role of a reservoir which absorbs the energy and momentum of the ions; as a result of collision with the atoms the ion distribution function relaxes to the Maxwellian distribution with the electrode temperature.

Neglecting Coulomb interaction between ions, we write the kinetic equation in the form

$$v \frac{\partial f}{\partial x} + \frac{eE}{M} \frac{\partial f}{\partial v} = \frac{n f_0 - f}{\tau} \tag{1.1}$$

$$\left(n = n(x) = \int_{-\infty}^{\infty} f(v, x)\, dv \right)$$

$$f_0 = \frac{1}{v_0 \sqrt{\pi}} \exp \frac{-Mv^2}{2T} \qquad \left(v_0^2 = \frac{2T}{M} \right) \tag{1.2}$$

Here $f = f(v, x)$ is the ion distribution function, x is the coordinate orthogonal to the electrode, v is the velocity component along the axis, E is the electric field intensity, M is the mass of the atom, and τ is the distribution function relaxation time.

Since the electron distribution is in equilibrium, their density, which because of quasi-neutrality coincides with the plasma density n, is connected with the electric field by the barometric formula

$$E = -\frac{T_e}{e} \frac{1}{n} \frac{dn}{dx} \tag{1.3}$$

Thus, (1.1) is nonlinear. We assume the electric field small and linearize (1.1) by replacing the distribution function $f(v, x)$ in the nonlinear term by its equilibrium value $n(x) f_0(v)$. As a result we obtain

$$\tau v \frac{\partial f}{\partial x} + f = n f_0 - \tau \alpha v \frac{dn}{dx} f_0 \qquad \left(\alpha = \frac{T_e}{T} \right) \tag{1.4}$$

We introduce the function

$$\psi(u, \xi) = \vartheta_0 f(-v, x) + \alpha v_0 n(x) f_0(v) \left(u = -\frac{v}{v_0}, \quad \xi = \frac{x}{\tau v_0} \right) \tag{1.5}$$

Then (1.4) can be written in the form

$$-u \frac{\partial \psi}{\partial \xi} + \psi = \beta n \frac{e^{-u^2}}{\sqrt{\pi}} \qquad \left(\int_{-\infty}^{\infty} \psi(u, \xi)\, du = \beta n(\xi), \quad \beta = 1 + \alpha \right) \tag{1.6}$$

We note that an equation analogous to (1.6) was studied in [1] in connection with the calculation of the slip coefficient of a rarefied gas.

2. We seek the solution of (1.6) in the region $\xi > 0$. Fourier-transforming with respect to ξ, we obtain

$$\Psi(u, k) = \frac{1}{1 + iku} \left(\frac{1}{\sqrt{\pi}} e^{-u^2} N(k) - \frac{u}{\sqrt{2\pi}} \psi(u, 0) \right) \tag{2.1}$$

where $\Psi(u, k)$ and $N(k)$ are the Fourier transforms of the functions $\psi(u, \xi)$ and $\beta n(\xi)$, respectively. We assume that $\psi(u, \xi)$ grows algebraically as $\xi \to \infty$ and therefore (2.1) holds for $\operatorname{Im} k > 0$. Since in this region $\Psi(u, k)$ must be an analytic function, (2.1) implies that for $k = i/u$ $(u > 0)$ the following equality holds:

$$\frac{e^{-u^2}}{\sqrt{\pi}} N\left(\frac{i}{u} \right) = \frac{u}{\sqrt{2\pi}} \psi(u, 0) \tag{2.2}$$

Integrating (2.1) with respect to u and introducing the new complex variable $\eta = i/k$, we obtain

$$N\left(\frac{i}{\eta} \right) = N\left(\frac{i}{\eta} \right) \frac{\eta}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{\eta - t} dt - \frac{\eta}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{t \psi(t, 0)}{\eta - t} dt \tag{2.3}$$

Equality (2.3) is valid in the region Re $\eta \equiv u > 0$. Passing to the limit in (2.3) to the real axis in accordance with the equality

$$\lim \frac{1}{t-\eta} = P\frac{1}{t-u} \pm \pi i \delta(t-u), \qquad \operatorname{Im} \eta \to 0$$

and using (2.2), we obtain

$$\psi(u, 0)\lambda(u) = \frac{e^{-u^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{t\psi(t, 0)}{t-u} dt \quad \left(\lambda(u) = 1 + \frac{u}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-u} dt\right) \tag{2.4}$$

Replacing in (2.4) the function ψ (u, 0) in accordance with (1.5) through the distribution function f (v, 0) and introducing

$$f(u) = v_0 f_0(-v, 0) \tag{2.5}$$

we obtain the nonhomogeneous singular integral equation for the function $f(u)$ for $0 \leq u \leq \infty$:

$$\lambda(u) f(u) = \frac{e^{-u^2}}{\sqrt{\pi}} \int_0^{\infty} \frac{tf(t)}{t-u} dt + \frac{e^{-u^2}}{\sqrt{\pi}} g(u) \quad \left(g(u) = \int_0^{\infty} \frac{tf(-t)}{t+u} dt\right) \tag{2.6}$$

We note that according to (2.5)

$$\int_{-\infty}^{\infty} f(u)\, du = n(0) \tag{2.7}$$

For negative values of the argument the function f (u) describes the incoming particles and is defined by the boundary conditions. Specifically, in the problem in question there is no incoming flux and therefore $g(u) \equiv 0$.

3. For the solution of (2.6) when $g(u) \equiv 0$, we introduce the function

$$\Phi(\eta) = \frac{1}{2} \int_0^{\infty} \frac{tf(t)}{t-\eta} dt \tag{3.1}$$

For the limiting values of this function above Φ_+ and below Φ_- the branch cut line $[0, \infty)$, we have the following relations [2]:

$$\Phi_+(u) - \Phi_-(u) = \pi i u f(u), \quad \Phi_+(u) + \Phi_-(u) = \int_0^{\infty} \frac{tf(t)}{t-u} dt \tag{3.2}$$

Using (3.2), we write the singular integral equation (2.6) in the form

$$\Phi_+ = \frac{\Lambda_+}{\Lambda_-} \Phi_- \tag{3.3}$$

where the distribution function is expressed through the limiting values of the function Φ on the real axis,

$$f(u) = \frac{2e^{-u^2}}{\sqrt{\pi}} \frac{\Phi_+}{\Lambda_-}, \qquad \Lambda_{\pm}(u) = \lambda(u) \pm i\sqrt{\pi}\, u e^{-u^2} \tag{3.4}$$

The functions $\Lambda_{\pm}(u)$ can be considered as the limiting values of the function

$$\Lambda(\eta) = 1 + \frac{\eta}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-\eta} dt \tag{3.5}$$

respectively, above and below the branch cut line $(-\infty < u < \infty)$. We see from (3.5), (3.4), and (2.4) that

$$\Lambda(\eta) = \Lambda(-\eta), \qquad \Lambda_+(u)^* = \Lambda_-(u) \tag{3.6}$$

The function

$$\mathrm{Ln}\,\frac{\Lambda_+}{\Lambda_-} = \mathrm{Arg}\,\frac{\Lambda_+}{\Lambda_-} = 2i\,\mathrm{Arc\,tg}\,\frac{\sqrt{\pi}\,ue^{-u^2}}{\lambda(u)} + 2i\pi m \tag{3.7}$$

(we select that branch of Arc tg which equals zero for u = 0) satisfies the Holder condition on the interval $[0, \infty)$, and therefore the solution of (3.3) can be written in the form

$$\Phi(\eta) = c\,\frac{e^{\Gamma(\eta)}}{\eta^k}, \quad \Gamma(\eta) = \frac{1}{2\pi i}\int_0^\infty \frac{dt}{t-\eta}\left(\ln\frac{\Lambda_+}{\Lambda_-} + 2m\pi i\right) \tag{3.8}$$
$$(k = 0, \pm 1, \ldots)$$

We note that

$$\lambda(u) = 1 - \sqrt{\pi}\,uv(u) \tag{3.9}$$

where the function v(u) is tabulated in [3]. Using tables of the function v(u), we can show that $\lambda(u)$ vanishes only at one point of the interval $[0, \infty)$. For u = 0, $\lambda(u) = 1$, and we see from (2.4) that as $u \to \infty$

$$\lambda(u) = -\frac{1}{2u^2} + O\left(\frac{1}{u^3}\right) \tag{3.10}$$

Thus, when u runs through values from 0 to ∞, the function

$$\ln\frac{\Lambda_+}{\Lambda_-} = 2i\,\mathrm{Arc\,tg}\,\frac{\sqrt{\pi}\,ue^{-u^2}}{\lambda(u)}$$

varies in the interval from zero to $2\pi i$. Therefore, in order that integral (3.8) exist it is necessary to set m = −1. As a result we obtain

$$\Gamma(\eta) = -\int_0^\infty \left(1 - \frac{1}{\pi}\,\mathrm{Arc\,tg}\,\frac{\sqrt{\pi}\,te^{-t^2}}{\lambda(t)}\right)\frac{dt}{t-\eta} \tag{3.11}$$

To find the exponent k, in (3.8) we expand $\Phi(\eta)$ in powers of $1/\eta$. From (3.1) follows

$$\Phi(\eta) = -\frac{\langle u\rangle}{2\eta} - \frac{\langle u^2\rangle}{2\eta^2} - \frac{\langle u^3\rangle}{2\eta^3} - \ldots \tag{3.12}$$

where $\langle u^k\rangle$ is the moment of the distribution function $f(u)$

$$\langle u^k\rangle = \int_0^\infty f(u)\,u^k\,du \tag{3.13}$$

We denote the first moment (k = 1) by $\langle u\rangle$. It follows from (2.5) that the ion flux leaving the plasma is

$$I = -\langle u\rangle v_0 \tag{3.14}$$

On the other hand, as $\eta \to \infty$, $\exp\Gamma(\eta) \to 1$ and therefore in (3.8) we must set

$$k = 1, \qquad c = -1/2\langle u\rangle \tag{3.15}$$

We shall show that the function $\Phi(\eta)$ remains bounded for $\eta = 0$. In fact, as $\eta \to 0$ we obtain from (3.11)

$$\Gamma(\eta) = \frac{1}{\pi}\int_A^\infty \mathrm{Arc\,tg}\,\frac{\sqrt{\pi}\,ue^{-u^2}}{\lambda(u)}\frac{dt}{t-\eta} + \frac{1}{\pi}\int_0^A \mathrm{Arc\,tg}\,\frac{\sqrt{\pi}\,ue^{-u^2}}{\lambda(u)}\frac{dt}{t-\eta} - \ln\frac{A-\eta}{\eta} \to \mathrm{const} + \ln\eta \tag{3.16}$$

and, therefore, for $\eta = 0$ $\Phi(\eta)$ is bounded and not equal to zero. The same may be said of the value of the distribution function $f(u)$ for u = 0. Comparing (3.8) and (3.15), we obtain

$$\Phi(\eta) = \frac{\langle u\rangle}{2}X(\eta) \qquad \left(X(\eta) = -\frac{e^{\Gamma(\eta)}}{\eta}\right) \tag{3.17}$$

Hence in accordance with (3.4) we obtain

$$f(u) = \frac{\langle u \rangle}{\sqrt{\pi}} e^{-u^2} \frac{X_+(u)}{\Lambda_+(u)} \tag{3.18}$$

The functions $\Gamma(\eta)$ and $\Lambda(\eta)$ are given by (3.11) and (3.5), respectively.

4. In principle, (3.18) represents the sought ion distribution function for $\xi = 0$. However, the function $X_+(u)$ contains an integral in the sense of the principal value, obtained by the limit passage in (3.11) to the real axis, which makes it difficult to analyze the result. We shall show that the solution may be written in a different form, in which this difficulty is eliminated.

Setting m = −1 in (3.8) and considering that

$$\Lambda_+(t) = \Lambda_-(-t) \tag{4.1}$$

we write

$$2\pi i\{\Gamma(\eta) + \Gamma(-\eta)\} = \int_0^\infty \frac{\ln \Lambda_+(t) - \pi i}{t-\eta} dt + \int_{-\infty}^0 \frac{\ln \Lambda_+(t) + \pi i}{t-\eta} dt - \int_0^\infty \frac{\ln \Lambda_-(t) + \pi i}{t-\eta} dt - \int_{-\infty}^0 \frac{\ln \Lambda_- - \pi i}{t-\eta} dt \tag{4.2}$$

We examine the two auxiliary functions

$$P(\eta) = \Lambda(\eta) \frac{\eta^2}{|\eta|^2} e^{-\pi i}, \quad Q(\eta) = \Lambda(\eta) \frac{\eta^2}{|\eta|^2} e^{\pi i} \tag{4.3}$$

the first of which we define in the upper half-plane of the complex variable η and the second in the lower half-plane. Since the logarithm of a real positive number is real, we obtain

$$\ln P_+(u) = \ln \Lambda_+(u) - \pi i, \quad \ln Q_-(u) = \ln \Lambda_-(u) + \pi i \quad (u > 0) \tag{4.4}$$

$$\ln P_+(u) = \ln \Lambda_+(u) + \pi i, \quad \ln Q_-(u) = \ln \Lambda_-(u) - \pi i \quad (u < 0) \tag{4.5}$$

Taking account of (4.4) and (4.5), we write (4.2) in the form

$$2\pi i\{\Gamma(\eta) - \Gamma(-\eta)\} = \int_{-\infty}^\infty \frac{\ln P_+(t)}{t-\eta} dt - \int_{-\infty}^\infty \frac{\ln Q_-(t)}{t-\theta} dt \tag{4.6}$$

Further, since

$$\ln P_+(t) = \ln \{2\Lambda_+ t^2 e^{-\pi i}\} - \ln 2|t|^2$$
$$\ln Q_-(t) = \ln \{2\Lambda_- t^2 e^{-\pi i}\} - \ln 2|t|^2$$

we obtain from (4.6)

$$2\pi i\{\Gamma(\eta) + \Gamma(-\eta)\} = \int_{-\infty}^\infty \frac{\ln(2\Lambda_+ t^2 e^{-\pi i})}{t-\eta} dt - \int_{-\infty}^\infty \frac{\ln(2\Lambda - t^2 e^{\pi i})}{t-\eta} dt \tag{4.7}$$

We calculate the Cauchy integral of the function

$$\ln[2\Lambda(\eta)\eta^2 e^{-\pi i}] \tag{4.8}$$

along the contour shown in Fig. 1. According to (3.5)

$$\Lambda(\eta) \approx -1/2\eta^2, \quad \eta \to \infty$$

and therefore on the circle of large radius

$$\ln[2\Lambda(\eta)\eta^2 e^{-\pi i}] \sim 0$$

As a result the integral over this part of the contour disappears. The integral along the semicircle around the point $\eta = 0$ also vanishes when the radius of the circle approaches zero. Thus the integral along the contour in question coincides with the integral along the real axis. On the other hand, it may be shown

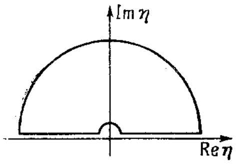

Fig. 1

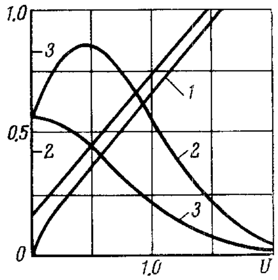

Fig. 2. Ion distribution function at boundary with electrode: 1) ratio of ion distribution function to the Maxwell function (scale on the right); 2) ion distribution function (scale on the left); 3) Maxwell function (scale on the left).

that in going around the contour in question the hodograph of the function $\Lambda(\eta)$ does not enclose the coordinate origin and, consequently, $\Lambda(\eta)$ does not have zeros inside the contour. This implies that function (4.8) is analytic inside the contour and

$$\int_{-\infty}^{\infty} \frac{\ln(2\Lambda_+ t^2 e^{-\pi i})}{t-\eta}\, dt = \begin{cases} 0 & \text{for } \operatorname{Im}\eta < 0 \\ 2\pi i \ln[2\Lambda(\eta)\eta^2 e^{-\pi i}] & \text{for } \operatorname{Im}\eta > 0 \end{cases} \tag{4.9}$$

Using similar reasoning, we can calculate the second integral on the right-hand side of (4.7). We obtain

$$[\Gamma(\eta) + \Gamma(-\eta)] = \begin{cases} \ln[2\Lambda(\eta)\eta^2 e^{-\pi i}] & \text{for } \operatorname{Im}\eta > 0 \\ \ln[2\Lambda(\eta)\eta^2 e^{\pi i}] & \text{for } \operatorname{Im}\eta < 0 \end{cases} \tag{4.10}$$

Hence

$$X(\eta) X(-\eta) = 2\Lambda(\eta) \tag{4.11}$$

where the function $X(\eta)$ is defined in (3.17).

Using (4.11), we can exclude $X(\eta)$ from (3.17) and write (3.18) in the form

$$f(u) = \frac{2\langle u\rangle}{\sqrt{\pi}} e^{-u^2} \frac{1}{X(-u)} \tag{4.12}$$

Since $u > 0$, this form of the sought distribution function does not contain integrals in the sense of the principal value.

5. Let us examine the behavior of the distribution function $f(u)$ as $u \to 0$ and $u \to \infty$. It follows from (2.6) and (2.7) that

$$f(0) = \frac{n(0)}{\sqrt{\pi}} \tag{5.1}$$

and using (4.12) we obtain

$$\langle u\rangle = \frac{n(0)}{\sqrt{2}} \tag{5.2}$$

Setting $\eta = 0$ in (4.11), we find that $X(0) = \sqrt{2}$. Thus

$$\lim \frac{f(u)}{\pi^{-1/2} n(0) \exp(-u^2)} = 1, \quad u \to 0$$

Differentiating (4.12), we can show that the derivative of the ion distribution function as $u \to 0$ becomes $+\infty$ logarithmically, i.e., in contrast with the Maxwellian function the maximum of the ion distribution function is shifted to the right. According to (3.11), as $u \to \infty$

$$\Gamma(-u) \approx \frac{l_0}{u}, \quad l_0 = \int_0^{\infty} \left\{1 - \frac{1}{\pi} \operatorname{Arc\,tg} \frac{\sqrt{\pi}\, t e^{-t^2}}{\lambda(t)}\right\} dt$$

and with account for (4.12) we find

$$f(u) \approx \frac{2\langle u\rangle}{\sqrt{\pi}} (u + l_0) e^{-u^2} \tag{5.3}$$

The value of l_0 obtained by numerical integration was 1.016.

Thus, for sufficiently large values of u

$$\frac{f(u)}{\pi^{-1/2} n(0) \exp(-u^2)} \approx \sqrt{2}(u + l_0)$$

i.e., the ion distribution function is enriched with fast particles. Figure 2 shows the results of the numerical calculation of the function $\sqrt{2}/X(-u)$, equal to the ratio of the ion distribution function to the Maxwellian distribution with density n(0) and temperature T (curve 1). The computation error is less than 0.1%. We see from Fig. 2 that the difference between the function $1/X(-u)$ and the asymptotic value $u + l_0$ is not large, and therefore the formula describes the distribution function well over practically the entire interval of velocities $u > 0$. This figure also shows the curve of the ion distribution function when normalized to unit density (curve 2),

$$\frac{1}{n(0)} f(u) = \left(\frac{2}{\pi}\right)^{1/2} e^{-u^2} \frac{1}{X(-u)}$$

Also shown for comparison is the Maxwell distribution $\pi^{-1/2} \exp(-u^2)$.

Let us calculate the moments of the distribution function. Formula (5.2) together with (3.14) makes it possible to find the connection between the ion flux and plasma density at the boundary with the electrode:

$$I = -\frac{n(0)}{\sqrt{2}} v_0 = -\frac{1}{4} n(0) \langle v \rangle \sqrt{2\pi} \left(\langle v \rangle = \left[\frac{8T}{\pi M}\right]^{1/2} \right) \qquad (5.4)$$

Thus the magnitude of the flux for a given plasma density is larger by about a factor of 2.5 than the flux calculated under the Maxwell distribution assumption. Using the distribution function (5.3), we find $I = -0.75 n(0) v_0$, which is very close to (5.4). Expanding the function $X(\eta)$ in powers of $1/\eta$, we obtain

$$X(\eta) = -\frac{1}{\eta}\left(1 + \frac{l_0}{\eta} + \frac{l_1 + l_0^2/\eta}{\eta^2} + \ldots\right)$$
$$l_1 = -\int_0^\infty \left(1 - \frac{1}{\pi} \operatorname{Arc\,tg} \frac{\sqrt{\pi}\, t e^{-t^2}}{\lambda(t)}\right) t\, dt \qquad (5.5)$$

Substituting expansions (3.12) and (5.5) into (3.17) and equating coefficients of like powers of $1/\eta$, we find

$$\langle u^2 \rangle = l_0 \langle u \rangle, \quad \langle u^3 \rangle = \left(\frac{l_0^2}{2} + l_1\right) \langle u \rangle$$

The value of l_1 equals 0.749. The mean kinetic energy corresponding to the x-component of the velocity is expressed through $\langle u^2 \rangle$ and equals

$$\frac{T}{n(0)} \langle u^2 \rangle = \frac{l_0}{\sqrt{2}} T$$

Thus, the mean ion energy near the electrode is about 0.7T, while at large distances it equals T/2. The mean energy carried by the ion to the electrode is

$$\left(1 + \frac{v_0}{|I|} \langle u^3 \rangle\right) T = \left(1 + l_1 + \frac{l_0^2}{2}\right) T$$

which yields 2.27T in place of 2T in the Maxwell distribution case.

The numerical calculations of the coefficients l_0 and l_1 and the function $X(-u)$ were made on an M-20 computer.

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